On the Okounkov-Olshanski Formula for the Number of Tableaux of Skew Shapes

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Partitions

- Partition: way to write a nonnegative integer as a sum of positive integers, without regard to order

$$9 = 3 + 3 + 2 + 1 \qquad \lambda = (3, 3, 2, 1) \qquad |\lambda| = 9$$

- Young diagram: grid of boxes representing a partition

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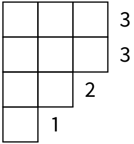
$[\lambda] =$ 

Standard Young Tableaux

- Standard Young tableau (SYT): way to fill in boxes of a Young diagram with 1 through $|\lambda|$, with rows and columns strictly increasing

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- f^λ: number of SYT of shape λ
- f^λ is the dimension of an irreducible representation of the symmetric group

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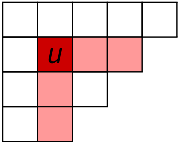
Hook-Length Formula

Theorem (Frame-Robinson-Thrall, 1954)

$$f^\lambda = \frac{|\lambda|!}{\prod_{u \in [\lambda]} h(u)}$$

- $h(u)$ is the size of a hook of a cell u



$h(u) = 5$

Applications of the Hook-Length Formula

- Hook lengths of (n, n) are

$$\begin{array}{|c|c|}\hline n+1 & n \\ \hline n & n-1 \\ \hline \end{array} \begin{array}{c} \cdots \\ \cdots \end{array} \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

so

$$f^{(n,n)} = \frac{(2n)!}{\prod_{u \in [(n,n)]} h(u)} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = C_n$$

- There exists a bijection between SYT of shape (n, n) and Dyck paths

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Standard Young Tableaux of Skew Shape

- Consider μ so that $[\mu] \subseteq [\lambda]$
  Example: take $\lambda = (3, 3, 2, 1)$, $\mu = (2, 1)$

- $f^{\lambda/\mu}$: number of SYT of shape λ/μ
- $f^{\lambda/\mu}$ are dimensions of irreducible representations of Hecke algebras
- Is there a formula for $f^{\lambda/\mu}$?

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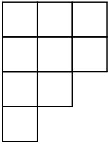

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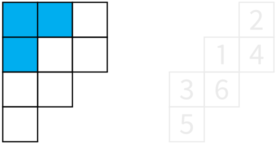

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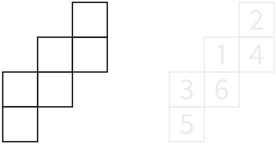

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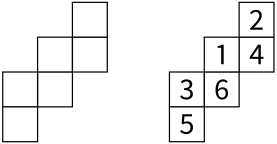

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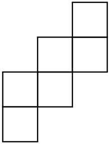
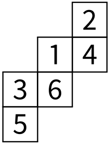
$[\lambda/\mu] =$ 



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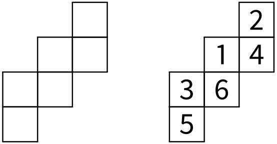
$[\lambda/\mu] =$ 

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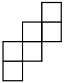
f  $= 61$

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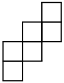
f  $= 61$

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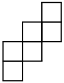
f  $= 61$

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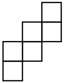
f  $= 61$

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The Okounkov-Olshanski Formula

Theorem (Okounkov-Olshanski, 1996)

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in[\lambda]} h(u)} \sum_{T\in \mathrm{SSYT}(\mu,d)} \prod_{(i,j)\in[\mu]} (\lambda_{d+1-T(i,j)} + i - j)$$

- $\mathrm{SSYT}(\mu, d)$ is *semistandard* tableaux of shape μ with all entries at most d

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$$f^{\ydiagram{2+2,3}} = \frac{5!}{5\cdot 4\cdot 3\cdot 1\cdot 3\cdot 2\cdot 1}\Big(\underbrace{3\cdot 2}_{\ytableaushort{11}} + \underbrace{3\cdot 3}_{\ytableaushort{12}} + \underbrace{4\cdot 3}_{\ytableaushort{22}}\Big)$$

$$= \frac{1}{3}\cdot 27 = 9$$

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Our Result

Theorem (Morales-Z., 2018+)

$$T(\lambda/\mu) = \det\left[\binom{\lambda_i - \mu_j + j - 1}{i - 1}\right]_{i,j=1}^{d}$$

$T((4,3)/(2)) = 3$ by previous example

$$\det\begin{bmatrix}\binom{2}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{4}{1}\end{bmatrix} = 1 \cdot 4 - 1 \cdot 1 = 3$$

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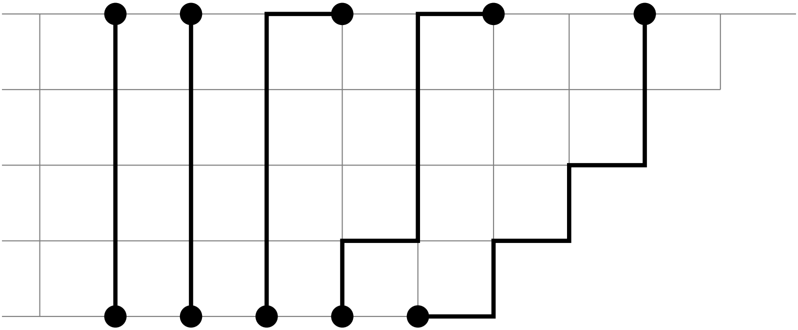

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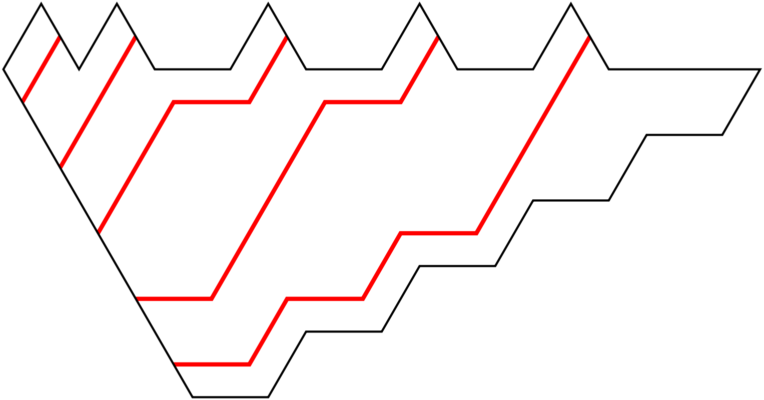

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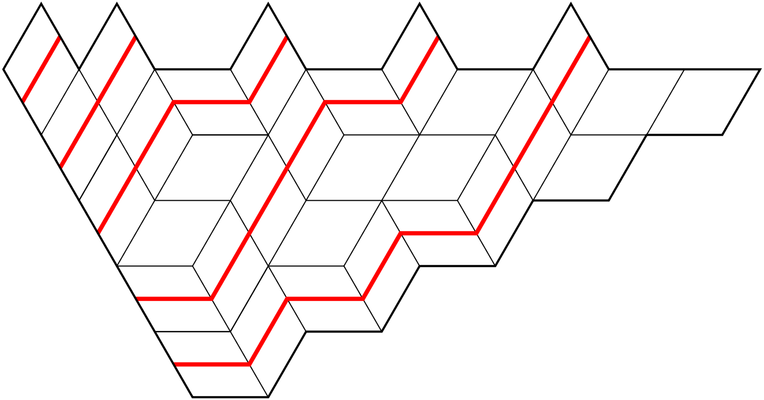

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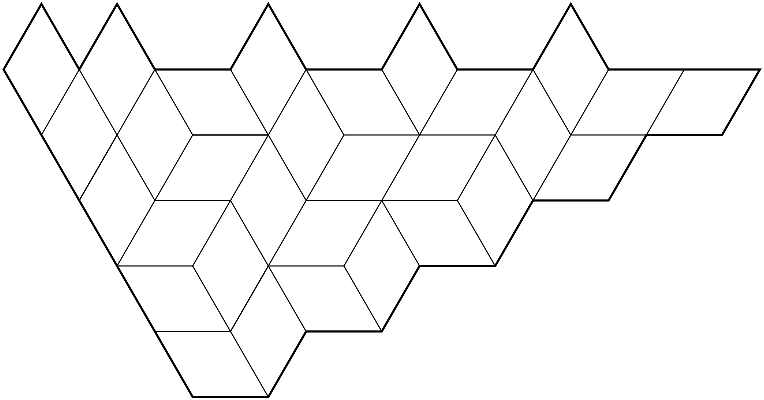

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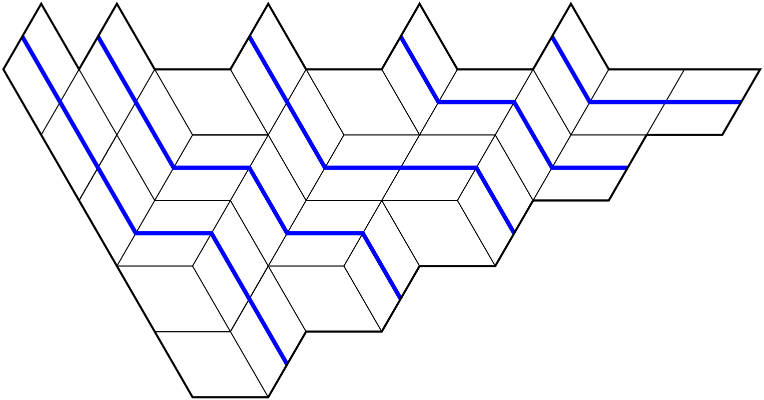

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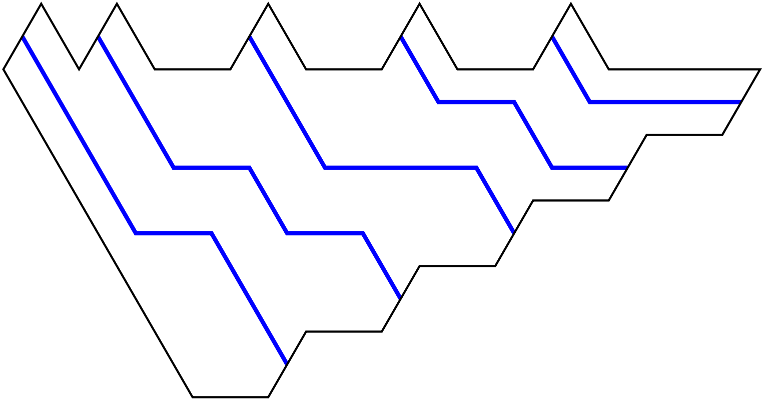

Reverse Plane Partitions

- Reverse plane partition (RPP): nonnegative integers, rows and columns weakly increasing

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- RPP(λ/μ): RPP of shape λ/μ

An RPP Generating Function

Definition

For $T \in \mathrm{RPP}(\lambda/\mu)$, let $|T|$ be the sum of entries in T. Then let

$$\mathrm{rpp}_{\lambda/\mu}(q) = \sum_{T \in \mathrm{RPP}(\lambda/\mu)} q^{|T|}.$$

Theorem (Stanley, 1971)

$$\lim_{q \to 1} \mathrm{rpp}_{\lambda/\mu}(q) \cdot (1-q)^{|\lambda/\mu|} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}$$

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where

- $w(u,t) = \lambda_{d+1-t} + i - j$ *where* $u = (i,j)$
- $p(T) = \sum_{u \in [\mu], m_T(u) \le t < T(u)} w(u,t)$
- $m_T(u)$ *is the minimum positive integer* t *such that replacing the entry of* u *in* T *with* t *still yields a semistandard tableau*

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- $p(T)$ *is a sum of* $w(u,t)$ *for some* u, t

Chen and Stanley have a similar q-analogue for semistandard tableaux

Other Work

Other results:

- Two other determinant formulas for $T(\lambda/\mu)$
- Equivalence of formulas for $f^{\lambda/\mu}$ of Okounkov-Olshanski and Knutson-Tao

Future work:

- Prove q-analogue without equivariant K-theory, possibly combinatorially
- Relate Okounkov-Olshanski to other formulas

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Acknowledgements

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- Dr. Tanya Khovanova
- MIT Math Department
- My family

Summary

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